On Q-Fuzzy Normal Subgroups

T.Priya[1], T.Ramachandran[2], K.T.Nagalakshmi[3]
[1] Department of Mathematics, PSNA College of Engineering and Technology, Dindigul-624 622, Tamilnadu , India.
[2]Department of Mathematics, M.V.Muthiah Government Arts College for Women, Dindigul-624001,Tamilnadu , India.
[3]Department of Mathematics, K L N College of Information and Technology, pottapalayam- 630611, Sivagangai District,Tamilnadu , India.

Abstract

In this paper, we introduce the concept of Q-fuzzy normal group and Q-fuzzy left (right) cosets and discussed some of its properties.

Keywords

Fuzzy group, Q-fuzzy group, Q-fuzzy normal group, Q-fuzzy normalizer, Q-fuzzy left (right) cosets.

AMS Subject Classification (2000): 20N25, 03E72, 03F055 , 06F35, 03G25.

1.Introduction

The concept of fuzzy sets was initiated by Zadeh**[20].** Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld **[13]** gave the idea of fuzzy subgroups.. A.Solairaju and R.Nagarajan **[16,17]** introduce and define a new algebraic structure of Q-fuzzy groups. In this paper we define a new algebraic structure of Q-fuzzy normal subgroups and Q-fuzzy left (right) cosets and study some of their properties.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

2.1 Definition [20]

Let X be any non empty set. A fuzzy subset μ of X is a function $\mu : X \rightarrow [0,1]$.

2.2 Definition [1,2]

Let G be a group. A fuzzy subset μ of G is called a fuzzy subgroup if for all $x, y \in G$,

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) = \mu(x)$

2.3 Definition [6,18]

Let G be a group. A fuzzy subgroup μ of G is said to be normal if for all $x, y \in G$,
$\mu(xyx^{-1}) = \mu(y)$ or $\mu(xy) = \mu(yx)$.

2.4 Definition[3]

Let μ be a fuzzy subgroup of a group G. For any $t \in [0,1]$, we define the level subset of μ is the set, $\mu^t = \{ x \in G, / \mu(x) \geq t \}$.

2.5 Definition [17]

Let Q and G be any two sets. A mapping $\mu : G \times Q \rightarrow [0, 1]$ is called a Q-fuzzy set in G.

2.6 Definition [10]

A Q-fuzzy set μ is called a Q-fuzzy subgroup of a group G, if for $x, y \in G$, $q \in Q$,

(i) $\mu(xy, q) \geq \min\{\mu(x,q), \mu(y,q)\}$

(ii) $\mu(x^{-1}, q) = \mu(x, q)$

3.Q-Fuzzy Normal Subgroups

3.1 Definition

Let G be a group. A Q- fuzzy subgroup μ of G is said to be normal if for all $x,y \in G$ and $q \in Q$, $\mu(xyx^{-1},q) = \mu(y, q)$ or $\mu(xy, q) \geq \mu(yx, q)$.

3.2 Definition

Let μ be a Q-fuzzy subgroup of a group G. For any $t \in [0,1]$, we define the level subset of μ is the set, $\mu^t = \{ x \in G, q \in Q \ / \mu(x, q) \geq t \}$.

Theorem 3.1

Let G be a group and μ be a Q-fuzzy subset of G. Then μ is a Q – fuzzy subgroup of G iff the level subsets, μ^t , $t \in [0,1]$, are subgroup of G.

Proof

Let μ be a Q-fuzzy subgroup of G and the level subset $\mu^t = \{x \in G, q \in Q / \mu(x,q) \geq t, t \in [0,1] \}$.

Let $x , y \in \mu^t$. Then $\mu(x , q) \geq t$ & $\mu(y , q) \geq t$.

Now $\mu(xy^{-1}, q) \geq \min\{ \mu(x , q) , \mu(y^{-1} , q) \}$

$= \min\{ \mu(x , q) , \mu(y , q) \}$

$\geq \min\{ t , t \}$

Therefore, $\mu(xy^{-1},q) \geq t$

This implies $xy^{-1} \in \mu^t$.

Thus μ^t is a subgroup of G.

Conversely, let us assume that μ^t be a subgroup of G.

Let $x , y \in \mu^t$. Then $\mu(x , q) \geq t$ and $\mu(y , q) \geq t$.

Also, $\mu(xy^{-1}, q) \geq t$, since $xy^{-1} \in \mu^t$

$= \min \{t , t\}$
$= \min \{ \mu (x , q) , \mu (y , q) \}$

That is, $\mu (xy^{-1}, q) \geq \min \{ \mu (x , q) , \mu (y , q) \}$

Hence μ is a Q-fuzzy subgroup of G.

3.3 Definition

Let G be a group and μ be a Q-fuzzy subgroup of G. Let $N(\mu) = \{ a \in G / \mu (axa^{-1}, q) = \mu (x ,q)$, for all $x \in G , q \in Q \}$. Then $N(\mu)$ is called the Q-fuzzy Normalizer of μ.

Theorem 3.2

Let G be a group and μ be a Q-fuzzy subset of G. Then μ is a Q- fuzzy normal subgroup of G iff the level subsets μ^t, $t\in[0,1]$, are normal subgroup of G.

Proof

Let μ be a Q- fuzzy normal subgroup of G and the level subsets μ^t , $t\in[0,1]$, is a subgroup of G. Let $x\in G$ and $a\in \mu^t$, then $\mu (a, q) \geq t$.

Now, $\mu (xax^{-1},q) = \mu (a, q) \geq t$,

since μ is a Q-fuzzy normal subgroup of G.

That is , $\mu (xax^{-1},q) \geq t$.

Therefore, $xax^{-1} \in \mu^t$.

Hence μ^t is a normal subgroup of G.

Theorem 3.3

Let μ be a Q-fuzzy subgroup of a group G.
Then
i. $N(\mu)$ is a subgroup of G.

ii. μ is a Q-fuzzy normal $\Leftrightarrow N(\mu) = G$.

iii. μ is a Q-fuzzy normal subgroup of the group $N(\mu)$.

Proof

i. Let $a , b \in N(\mu)$ then $\mu (axa^{-1} , q) = \mu (x , q)$, for all $x \in G$. $\mu (bxb^{-1} , q) = \mu (x , q)$, for all $x \in G$.

Now $\mu (abx(ab)^{-1} , q) = \mu (abxb^{-1}a^{-1} , q)$

$= \mu (bxb^{-1} , q)$

$= \mu (x , q)$

Thus we get, $\mu (abx(ab)^{-1} , q) = \mu (x , q)$.

$\Rightarrow ab \in N(\mu)$

Therefore, $N(\mu)$ is a subgroup of G.

ii. Clearly $N(\mu) \subseteq G$, μ is a Q-fuzzy normal subgroup of G.

Let $a \in G$, then $\mu (axa^{-1} , q) = \mu (x , q)$.

Then $a \in N(\mu) \Rightarrow G \subseteq N(\mu)$.

Hence $N(\mu) = G$.

Conversely, let $N(\mu) = G$.

Clearly $\mu (axa^{-1} , q) = \mu (x , q)$, for all $x \in G$ and $a \in G$.

Hence μ is a Q – fuzzy normal subgroup of G.

iii. From (2) , μ is a Q-fuzzy normal subgroup of a group $N(\mu)$.

3.4 Definition

Let μ be a Q-fuzzy subset of G. Let ${}_xf : G \times Q \to G \times Q$ [$f_x : G \times Q \to G \times Q$] be a function defined by ${}_xf(a , q) = (xa , q)$ [$f_x(a , q) = (ax , q)$]. A Q-fuzzy left (right) coset ${}_x\mu$ (μ_x) is defined to be ${}_xf(\mu)$ ($f_x(\mu)$).

It is easily seen that $({}_x\mu)(y , q) = \mu (x^{-1}y , q)$ and $(\mu_x)(y , q) = \mu(yx^{-1} , q)$, for every (y , q) in $G \times Q$.

Theorem 3.4 :

Let μ be a Q-fuzzy subset of G. Then the following conditions are equivalent for each x , y in G.

(i) $\mu (xyx^{-1} , q) \geq \mu (y , q)$

(ii) $\mu (xyx^{-1} , q) = \mu (y , q)$

(iii) $\mu (xy , q) = \mu (yx ,q)$

(iv) ${}_x\mu = \mu_x$

(v) ${}_x\mu_x^{-1} = \mu$

Proof : Straight forward.

Theorem 3.5 :

If μ is a Q-fuzzy subgroup of G, then $g\mu g^{-1}$ is also a Q-fuzzy subgroup of G for all $g \in G$ and $q\in Q$.

Proof :

Let μ be a Q-fuzzy subgroup of G.

Then (i) $(g\mu g^{-1}) (xy , q) = \mu (g^{-1}(xy)g , q)$

$= \mu(g^{-1}(xgg^{-1}y)g , q)$

$= \mu ((g^{-1}xg)(g^{-1}yg) , q)$

$\geq \min\{ \mu (g^{-1}xg, q) , \mu (g^{-1}yg , q)\}$

$\geq \min\{g\mu g^{-1}(x , q) , g\mu g^{-1}(y , q) \}$, for all x , y in G and $q \in Q$.

(ii) $g\mu g^{-1}(x , q) = \mu(g^{-1}xg , q)$

$= \mu ((g^{-1}xg)^{-1} , q)$

$= \mu (g^{-1}x^{-1}g , q)$

$= g\mu g^{-1}(x^{-1} , q)$, for all x , y in G and $q \in Q$.

Hence $g\mu g^{-1}$ is a Q-fuzzy subgroup of G.

Theorem 3.6 :

If μ is a Q-fuzzy normal subgroup of G, then $g\mu g^{-1}$ is also a Q-fuzzy normal subgroup of G , for all $g \in G$ and $q \in Q$.

Proof :

Let μ be a Q-fuzzy normal subgroup of G. then $g\mu g^{-1}$ is a subgroup of G.
Now

$$\begin{aligned} g\mu g^{-1}(xyx^{-1},q) &= \mu(g^{-1}(xy\, x^{-1})\, g, q) \\ &= \mu(xy\, x^{-1}, q) \\ &= \mu(y, q) \\ &= \mu(g\, y\, g^{-1}, q) \\ &= g\mu g^{-1}(y,q). \end{aligned}$$

Theorem 3.7 :

Let μ and λ be two Q-fuzzy subgroups of G. Then $\lambda\cap\mu$ is a Q-fuzzy subgroup of G.

Proof :

Let λ and μ be two Q-fuzzy subgroups of G

(i) $(\lambda\cap\mu)\,(xy^{-1}, q) = \min (\lambda\,(xy^{-1}, q)\, ,\mu\,(xy^{-1}, q))$
$\geq \min \{ \min\{\lambda(x , q) , \lambda\,(y ,q)\}, \min\{\mu(x , q) , \mu\,(y ,q)\}\}$

$\geq \min\{ \min\{\lambda(x , q) , \mu(x ,q)\}, \min\{\lambda(y, q),\mu(y ,q)\}\}$
$= \min \{ (\lambda\cap\mu)\,(x , q) , (\lambda\cap\mu)\,(y , q)\}$

Thus $(\lambda\cap\mu)\,(xy^{-1}, q) \geq \min\{(\lambda\cap\mu)(x, q),(\lambda\cap\mu)(y,q)\}$

(ii) $(\lambda\cap\mu)\,(x , q) = \{ \lambda\,(x , q) , \mu\,(x , q)\}$
$= \{ \lambda\,(x^{-1} , q) , \mu\,(x^{-1} , q)\}$
$= \{ (\lambda\cap\mu)\,(x^{-1} , q)\}$.

Hence $\lambda\cap\mu$ is a Q-fuzzy subgroup of G.

Remark : If μ_i , $i\in\Delta$ is a Q-fuzzy subgroup of G, then $\bigcap_{i\in\Delta} \mu_i$ is a Q-fuzzy subgroup of G.

Theorem 3.8 :

The intersection of any two Q-fuzzy normal subgroups of G is also a Q-fuzzy normal subgroup of G.

Proof :

Let λ and μ be two Q-fuzzy normal subgroups of G.
According to theorem 3.7, $\lambda\cap\mu$ is a Q-fuzzy subgroup of G.

Now for all x , y in G , we have

$(\lambda\cap\mu)\,(xyx^{-1} , q) = \min (\lambda(xyx^{-1} , q) , \mu(xyx^{-1} , q))$

$= \min (\lambda\,(y , q) , \mu(y , q))$

$=(\lambda\cap\mu)\,(y , q)$

Hence $\lambda\cap\mu$ is a Q-fuzzy normal subgroup of G.

Remark : If μ_i , $i\in\Delta$ are Q-fuzzy normal subgroup of G, then $\bigcap_{i\in\Delta} \mu_i$ is a Q-fuzzy normal subgroup of G.

Definition 3.5 :

The mapping f: G x Q → H x Q is said to be a group Q-homomorphism if

(i) f: G → H is a group homomorphism

(iii) $f(xy , q) = (f(x)f(y) , q)$, for all $x , y \in G$ and $q \in Q$.

Theorem 3.9 :

Let f: G x Q → H x Q is a group Q-homomorphism.

(i) If μ is a Q-fuzzy normal subgroup of H , then $f^{-1}(\mu)$ is a Q-fuzzy normal subgroup of G.

(ii) If f is an epimorphism and μ is a Q-fuzzy normal subgroup of G, then $f(\mu)$ is a Q-fuzzy normal subgroup of H.

Proof :

(i) Let f: G x Q → H x Q is a group Q-homomorphism and let μ be a Q-fuzzy Normal subgroup of H.

Now for all $x , y \in G$, we have

$$\begin{aligned} f^{-1}(\mu)(xyx^{-1}, q) &= \mu\,(f\,(xyx^{-1} , q)) \\ &= \mu\,(f(x)f(y)f(x)^{-1} , q) \\ &= \mu\,(f(y) , q) \\ &= f^{-1}(\mu)(y , q) \end{aligned}$$

Hence $f^{-1}(\mu)$ is a Q-fuzzy normal subgroup of G.

(ii) Let μ be a Q-fuzzy normal subgroup of G. Then $f(\mu)$ is a Q- fuzzy subgroup of H.

Now, for all u , v in H, we have

$$f(\mu)(uvu^{-1}, q) = \sup_{f(y)=uvu^{-1}} \mu\,(y , q) = \sup_{f(x)=u;\, f(y) = v} \mu\,(xyx^{-1}, q)$$

$$= \sup_{f(y) = v} \mu\,(y ,q) = f(\mu)(v , q)$$

(since f is an epimorphism)

Hence $f(\mu)$ is a Q-fuzzy normal subgroup of H.

Definition 3.6 :

Let λ and μ be two Q-fuzzy subsets of G. The product of λ and μ is defined to be the Q-fuzzy subset $\lambda\mu$ of G is given by
$\lambda\mu(x , q) = \sup \min (\lambda(y , q) , \mu(z , q)) , x \in G$.

$yz = x$

Theorem 3.10:

If λ & μ are Q-fuzzy normal subgroups of G , then $\lambda\mu$ is a Q-fuzzy normal subgroup of G .

Proof :

Let λ & μ be two Q-fuzzy normal subgroups of G.

(i) $\lambda\mu(xy, q) = \sup_{\substack{x_1y_1 = x \\ x_2y_2 = y}} \min\{\lambda(x_1y_1, q), \mu(x_2y_2, q)\}$

$\geq \sup_{\substack{x_1y_1 = x \\ x_2y_2 = y}} \min\{\min\{\lambda(x_1,q),\lambda(y_1,q)\}, \min\{\mu(x_2,q),\mu(y_2,q)\}\}$

$\geq \min\{\sup_{x_1y_1 = x}\min\{\lambda(x_1,q),\lambda(y_1, q)\}, \sup_{x_2y_2 = y}\min\{\mu(x_2, q), \mu(y_2, q)\}\}$

$\lambda\mu(xy, q) \geq \min\{\lambda\mu(x, q), \lambda\mu(y, q)\}$

(ii) $\lambda\mu(x^{-1}, q) = \sup_{(yz)^{-1} = x^{-1}} \min\{\mu(z^{-1}, q), \lambda(y^{-1}, q)\}$

$= \sup_{x = yz} \min\{\mu(z, q), \lambda(y, q)\}$

$= \sup_{x = yz} \min\{\lambda(y, q), \mu(z, q)\}$

$= \lambda\mu(x, q).$

Hence $\lambda\mu$ is a normal Q-fuzzy subgroup of G .

4. Conclusion

In this article we have discussed Q-fuzzy Normal subgroups, Q-fuzzy Normalizer and Q-fuzzy subgroups under homomorphism. Interestingly, It has been observed that Q-fuzzy concept adds an another dimension to the defined fuzzy normal subgroups. This concept can further be extended for new results.

References

[1] Biswas. R, Fuzzy subgroups and anti-fuzzy subgroups, Fuzzy sets and systems, 35(1990), 121-124.

[2] Choudhury.F.P. and Chakraborty.A.B. and Khare.S.S., A note on fuzzy subgroups and Fuzzy homomorphism .Journal of mathematical analysis and applications 131, 537 -553 (1988).

[3] Das. P.S, Fuzzy groups and level subgroups, J.Math.Anal. Appl, 84(1981) 264-269.

[4] Dixit.V.N., Rajesh Kumar, Naseem Ajmal., Level subgroups and union of fuzzy subgroups, Fuzzy Sets and Systems, 37, (1990),359-371.

[5] Malik D.S. and Mordeson J.N. Fuzzy subgroups of abelian groups, Chinese Journal of Math., vol.19 (2) , 1991.

[6] Mashour A.S, Ghanim M.H. and Sidky F.I. , Normal Fuzzy Subgroups,Univ.u Novom Sadu Zb.Rad.Prirod.-Mat.Fak.Ser.Mat. 20(2) ,1990, 53-59.

[7] Mehmet sait EROGLU, The homomorphic image of a fuzzy subgroup is always a Fuzzy subgroup, Fuzzy sets and Systems, 33(1989) 255 – 256.

[8] Mohamed Asaad, Groups and Fuzzy subgroups, Fuzzy sets and systems 39(1991) 323-328.

[9] Mustafa Akgul, Some properties of fuzzy groups, Journal of mathematical analysis and applications 133, 93-100 (1988).

[10] Muthuraj.R., Sithar Selvam.P.M., Muthuraman.M.S., Anti Q-fuzzy group and its lower Level subgroups, International journal of Computer Applications 3(3),2010,16-20.

[11] Prabir Bhattacharya, Fuzzy Subgroups: Some Characterizations,J.Math. Anal. Appl.128 (1987),241 – 252.

[12] Rajesh kumar, Homomorphism and fuzzy (fuzzy normal) subgroups, Fuzzy sets and Systems, 44(191) 165 – 168.

[13] Rosenfeld.A., fuzzy groups, J. math. Anal. Appl. 35 (1971),512-517.

[14] Samit kumar majumder and Sujit kumar sardar, On some properties of anti fuzzy subgroups, .Mech.cont.and Math.Sci.,3(2),2008,337-343.

[15] Sherwood, H : Product of fuzzy subgroups, Fuzzy sets and systems 11(1983) ,79-89.

[16] A.Solairaju and R.Nagarajan " Q- fuzzy left R- subgroups of near rings w.r.t T- norms", Antarctica journal of mathematics.vol 5,2008.

[17] A.Solairaju and R.Nagarajan , A New Structure and Construction of Q-Fuzzy Groups, Advances in Fuzzy mathematics, Vol 4 (1) (2009) pp.23-29.

[18] Vasntha Kandasamy.W.B., Smarandache Fuzzy Algebra, American Research Press,2003.

[19] Yunjie Zhand and Dong Yu , On fuzzy Abelian subgroups,

[20] Zadeh. L.A.Fuzzy sets, Information and Control,8(1965),338-353.